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Selection theorems for multivalued generalized contractions

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Abstract. The purpose of this paper is to report old and new selection results for multivalued operators. Main results of this paperconcern with the existence of a Caristi type selection for some multivalued generalized contractions on metric spaces.

1. Introduction

Caristi's fixed point theorem states that each operator f from a complete metric space (X,d) into itself satisfying the condition:

there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+$ such that:

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \quad \text{for each} \quad x \in X \tag{1}$$

has at least a fixed point $x^* \in X$, i.e. $x^* = f(x^*)$

Let (X,d) be a metric space and $\mathcal{P}(X)$ the space of all subsets of X. We denote by $P(X)$ the space of all nonempty subsets of X and by $P_p(X)$ the set of all nonempty subsets of X having the property "p", where "p" could be: cl=closed, b=bounded, cp=compact, ehcv=extremelly hyperconvex, cv=convex (for normed spaces X), dc=decomposable (for the case of L^1 spaces), etc.

We consider the following functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \inf\ \{\ d(a,b) \mid a \in A,\ b \in B\ \}$$

$$H : P_b(X) \times P_b(X) \to \mathbb{R}_+,\ H(A,B) = \max\ \{\ \sup_{a\in A} D(a,B),\ \sup_{b\in B} D(b,A)\ \}.$$

It is well-known that if (X,d) is a complete metric space, then $(P_{b,cl}(X), H)$ is also a complete metric space.

If X, Y are nonempty sets and $F : X \to P(X)$ is a multivalued operator then a selection of F is a singlevalued operator $f : X \to Y$ such that $f(x) \in F(x)$, for each $x \in X$.

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2. A review of known results

One of the most important problem in multivalued analysis is to establish sufficient conditions for the existence of a selection, with a certain regularity (measurable, continuous, Lipschitz, etc.), for a multivalued operator. The purpose of this section is to report several selection results in this respect.

Let us start with the case of measurable selections.

Theorem 2.1 (Kuratowski, Ryll-Nardzewski)**.** *Let $(T, \mathcal{A})$ be a measurable space, (X, d) be a complete separable metric space and $F : T \to P_{cl}(X)$ be weak measurable. Then there exists a measurable selection for F.*

If the multivalued operator is lower semi-continuous then we have:

Theorem 2.2 (Michael)**.** *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P_{cl,cv}(Y)$ be l.s.c. on X. Then there exists $f : X \to Y$ a continuous selection of F.*

If the multivalued operator has open fibres then Browder proved the following:

Theorem 2.3 (Browder)**.** *Let X and Y be Hausdorff topological vectorial space and $K \in P_{cp}(X)$. Let $F : K \to P_{cv}(Y)$ be a multivalued operator such that $F^{-1}(y)$ is open, for each $y \in Y$. Then there exists a continuous selection f of F.*

For the case of a multivalued operator with decomposable values we have:

Theorem 2.4 (A. Petruşel - G. Moţ, 2001)**.** *Let E be a Banach space such that $L^1(T, E)$ is separable. Let K be a nonempty, paracompact, decomposable subset of $L^1(T, E)$ and let $F : K \to P_{dec}(K)$ be a multi-valued operator such that $F^{-1}(y)$ is open, for each $y \in Y$. Then F has a continuous selection.*

Contrary to the lower semi-continuous case, for an upper semi-continuous operator we only have an approximate selection result:

Theorem 2.5 (Cellina)**.** *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P_{cv}(Y)$ be u.s.c. on X. Then for each $\varepsilon > 0$ there exists a continuous operator $f_\varepsilon : X \to Y$ such that $Graf\, f_\varepsilon \subset V(Graf\, F, \varepsilon)$.*

An interesting result for a multivalued operator on $[0, 1]$ with no necessary closed or convex values is the following:

Theorem 2.6 (Strother)**.** *Let $F : [0, 1] \to P([0, 1])$ be a continuous multivalued operator. Then there exists a continuous selection of F.*

Let us consider now the problem of the existence of a Lipschitz type selection. First we mention:

Theorem 2.7 (Hermes)**.** *Let $T > 0$ and $F : [0, T] \to P_{cp}(\tilde{B}(0; R)))$ an a-Lipschitz multifunction. Then there exists an a-Lipschitz selection of F.*

Moreover we have:

Theorem 2.8 (Yost)**.** *Let X be a metric space and Y be a Banach space. Then every a-Lipschitz multifunction $F : X \to P_{b,cl,cv}(Y)$ admits a Lipschitz selection if and only if Y is finite dimensional.*

For multivalued operators on hyperconvex metric spaces we have:

Theorem 2.9 (Kirk - Khamsi - Martinez, 2000)**.** *Let (X, d) be a hyperconvex metric space and $F : X \to P_{cl,ehcv}(X)$ be an a-Lipschitz multifunction. Then there exists an a-Lipschitz selection of F.*

3. Caristi selection for multivalued generalized contractions

First result of this type was established by J. Jachymski for a multivalued contraction with closed values.

Theorem 3.1 (J. Jachymski, 1998)**.** *Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued contraction. Then there exists $f : X \to X$ a Caristi selection (with a Lipschitz map φ) of F.*

An extension for a Reich type multivalued operator is the following:

Theorem 3.2 (A. Petruşel - A. Sîntămărian, 2002)**.** *Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a Reich type multivalued operator, i. e. there exist $a, b, c \in \mathbb{R}_+$, with $a + b + c < 1$ and for each $x, y \in X$*

$$H(F(x), F(y)) \leq a \cdot d(x, y) + b \cdot D(x, F(x)) + c \cdot D(y, F(y)).$$

Then there exists $f : X \to X$ a Caristi selection of F.

Then, another generalization of Jachymski's result was recently proved by Sîntămărian:

Theorem 3.3 (A. Sîntămărian, 2005)**.** *Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a generalized multivalued contraction, i. e. for each $x, y \in X$*
$H(F(x), F(y)) \leq a_1\ d(x, y) + a_2\ D(x, F(x)) + a_3\ D(y, F(y)) + a_4\ D(x, F(y)) + a_5\ D(y, F(x))$, where $a_1 + a_2 + a_3 + 2a_4 \in]0, 1[$.
Then there exists $f : X \to X$ a Caristi selection of F.

The first main result of this work is:

Theorem 3.4. *Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a Ciric type multivalued contraction, i. e. there is $q \in]0, 1[$ such that for each $x, y \in X$*
$H(F(x), F(y)) \leq q \cdot \max\{d(x, y), D(x, F(x)), D(y, F(y)), \frac{1}{2}(D(x, F(y)) + D(y, F(x)))\}$.
Then there exists $f : X \to X$ a Caristi selection of F.

Proof. Let $\varepsilon := \frac{1-q}{2}$ and $\varphi(x) := \frac{1}{\varepsilon} \cdot D(x, F(x))$.
Then, obviously $\varepsilon + q = \frac{1+q}{2} < 1$ and φ is bounded below by 0.
Since $\frac{1}{\varepsilon+q} > 1$, for each $x \in X$ we can choose $f(x) \in F(x)$ such that

$$d(x, f(x)) \leq \frac{1}{\varepsilon + q} \cdot D(f(x), F(x)),\ \text{for each}\ x \in X.$$

We have then successively:

$$
\begin{aligned}
&D(f(x),F(f(x))) \leq H(F(x),F(f(x))) \leq \\
&q\cdot \max\{d(x,f(x)),D(x,F(x)),D(f(x),F(f(x)),\frac{1}{2}(D(x,F(f(x)))+ \\
&\qquad\qquad\qquad\qquad + D(f(x),F(x)))\} \leq \\
&\leq q\cdot \max\{d(x,f(x)),d(x,f(x)),D(f(x),F(f(x))),\frac{1}{2}D(x,F(f(x)))\} = \\
&= q\cdot \max\{d(x,f(x)),D(f(x),F(f(x))),\frac{1}{2}D(x,F(f(x)))\}.
\end{aligned}
$$

1) If $\max\{d(x,f(x)),D(f(x),F(f(x))),\frac{1}{2}D(x,F(f(x)))\} = d(x,f(x))$ then we obtain: $D(f(x),F(f(x))) \leq q\cdot d(x,f(x)), x\in X$.
2) If $\max\{d(x,f(x)),D(f(x),F(f(x))),\frac{1}{2}D(x,F(f(x)))\} = D(f(x),F(f(x)))$ then $D(f(x),F(f(x))) \leq q\cdot D(f(x),F(f(x))), x\in X$, a contradiction with $q>1$.
3) If $\max\{d(x,f(x)),D(f(x),F(f(x))),\frac{1}{2}D(x,F(f(x)))\} = \frac{1}{2}D(x,F(f(x)))$ then $D(f(x),F(f(x))) \leq \frac{q}{2}\cdot D(x,F(f(x))) \leq \frac{q}{2}[d(x,f(x)+D(f(x),F(f(x))]$ and hence $D(f(x),F(f(x))) \leq \frac{q}{2-q}\cdot d(x,f(x)) \leq q\cdot d(x,f(x)), x\in X$.

Hence in all the three cases we have:

$$D(f(x),F(f(x))) \leq q\cdot d(x,f(x)), x\in X.$$

We will prove now that f is a Caristi type operator. Indeed, for each $x\in X$ we have:

$$
\begin{aligned}
d(x,f(x)) &= \frac{1}{\varepsilon}\cdot[(\varepsilon+q)\cdot d(x,f(x)) - q\cdot d(x,f(x))] \leq \\
&\leq \frac{1}{\varepsilon}[D(x,F(x)) - D(f(x),F(f(x)))] = \\
&= \varphi(x) - \varphi(f(x)).
\end{aligned}
$$

□

Remark 3.1. It is quite obvious that the above theorems includes as particular cases Theorem 3.1 - Theorem 3.3.

For the case of a multivalued contraction with variable coefficient, Xu proved:

Theorem 3.5 (Xu, 2002). *Let (X,d) be a metric space and $F: X\to P_{b,cl}(X)$ be a multivalued operator. Suppose there exists a lower semicontinuous mapping $\alpha: X\to [0,1[$ such that*

$$H(F(x),F(y)) \leq \alpha(x)\cdot d(x,y),\ \textit{for each } x,y\in X.$$

Then there exists $f: X\to X$ a Caristi selection (with a lower semicontinuous map φ) of F.

The second main result of the paper is:

Theorem 3.6. *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued operator. Suppose there exist the lower semicontinuous mappings $\alpha, \beta, \gamma : X \to \mathbb{R}_+$, with $\alpha(x)+\beta(x)+\gamma(x) < 1$ and for each $x \in X$, such that for each $x, y \in X$ we have:*

$$H(F(x), F(y)) \le \alpha(x) \cdot d(x,y) + \beta(x) \cdot D(x, F(x)) + \gamma(x) \cdot D(y, F(y)).$$

Then there exists $f : X \to X$ a Caristi selection of F.

Proof. Let $\varepsilon(x) := \frac{1-\alpha(x)-\beta(x)}{1-\gamma(x)}$ and $\varphi(x) := \frac{1}{\varepsilon(x)} \cdot D(x, F(x))$.

Then φ is bounded below by 0.

Note that $\frac{\alpha(x)+\beta(x)}{1-\gamma(x)} + \varepsilon(x) = \frac{1}{1-\gamma(x)} > 1$, for each $x \in X$.

Then there is $f(x) \in F(x)$ such that

$$d(x, f(x)) \le \frac{1}{1-\gamma(x)} \cdot D(f(x), F(x)), \text{ for each } x \in X.$$

Note that $D(f(x), F(f(x))) \le H(F(x), F(f(x))) \le$
$\alpha(x) \cdot d(x, f(x)) + \beta(x) \cdot D(x, F(x)) + \gamma(x) \cdot D(f(x), F(f(x))) \le \alpha(x) \cdot d(x, f(x)) +$
$\beta(x) \cdot d(x, f(x)) + \gamma(x) \cdot D(f(x), F(f(x)))$.

Hence $D(f(x), F(f(x))) \le \frac{\alpha(x)+\beta(x)}{1-\gamma(x)} \cdot d(x, f(x)), x \in X$.

It remains to show that f satisfies the Caristi type condition. For each $x \in X$ we have:

$d(x, f(x)) =$
$\frac{1}{\varepsilon(x)} \cdot [(\varepsilon(x) + \frac{\alpha(x)+\beta(x)}{1-\gamma(x)}) \cdot d(x, f(x)) - \frac{\alpha(x)+\beta(x)}{1-\gamma(x)} \cdot d(x, f(x))] \le$
$\frac{1}{\varepsilon(x)} \cdot [\frac{1}{1-\gamma(x)} \cdot d(x, f(x)) - D(f(x), F(f(x)))] \le$
$\frac{1}{\varepsilon(x)} [D(x, F(x)) - D(f(x), F(f(x)))] =$
$\varphi(x) - \varphi(f(x))$. □

In a similar way to the above results we have:

Theorem 3.7. *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$. Suppose there exists a lower semicontinuous mapping $q : X \to [0,1[$ such that for each $x, y \in X$*

$H(F(x), F(y)) \le q(x) \cdot \max\{d(x,y), D(x, F(x)), D(y, F(y)),$
$\frac{1}{2}(D(x, F(y)) + D(y, F(x)))\}$.

Then there exists $f : X \to X$ a Caristi selection of F.

4. Open questions

I.

Give other examples of generalized multivalued contractions having Caristi type selections.

II.

Let X be a nonempty set and $s(X) := \{(x_n)_{n \in \mathbb{N}} | x_n \in X,\ n \in \mathbb{N}\}$.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.
(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition an element of $c(X)$ is convergent sequence and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we write

$$x_n \to x \text{ as } n \to \infty.$$

In what follow we will denote an L-space by $(X, \to)$.

Example 4.1 (L-structures on ordered sets)**.** Let $(X, \leq)$ be an ordered set.

(a) $c_1(X) := \{(x_n)_{n\in\mathbb{N}} | (x_n)_{n\in\mathbb{N}}$ is increasing and there exists $\sup x_n\}$, $Lim(x_n)_{n\in\mathbb{N}} = \sup\{x_n | n \in \mathbb{N}\}$. If $x = \sup\{x_n | n \in \mathbb{N}\}$, $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence, then we denote this by $x_n \uparrow x$.
(b) $c_2(X) := \{(x_n)_{n\in\mathbb{N}} | (x_n)_{n\in\mathbb{N}}$ is decreasing and there exists $\inf\{x_n | n \in \mathbb{N}\}$, $Lim(x_n)_{n\in\mathbb{N}} = \inf\{x_n | n \in N\}$. If $(x_n)_{n\in\mathbb{N}}$ is decreasing and $\inf\{x_n | n \in \mathbb{N}\} = x$, then we denote this by $x_n \downarrow x$.
(c) $c(X) := c_1(X) \cup c_2(X)$. If $x = Lim(x_n)_{n\in\mathbb{N}}$, then we denote this by $x_n \overset{m}{\to} x$ as $n \to \infty$.
(d) By definition, a sequence $(x_n)_{n\in\mathbb{N}}$ (0)-converges to x if there exist two sequence $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that
  (i) $a_n \uparrow x$ and $b_n \downarrow x$;
  (ii) $a_n \leq x_n \leq b_n$, $n \in \mathbb{N}$.

We denote this convergence by $x_n \overset{0}{\to} x$. It is clear that $(X, \uparrow)$, $(X, \downarrow)$, $(X, \overset{m}{\to})$, $(X, \overset{0}{\to})$ are L-spaces.

Example 4.2 (L-structures on Banach spaces)**.** Let X be a Banach space. We denote by $\to$ the strong convergence in X and by $\rightharpoonup$ the weak convergence in X. Then $(X, \to)$, $(X, \rightharpoonup)$ are L-spaces.

Example 4.3 (L-structures on function spaces)**.** Let X and Y be two metric spaces. Let $\mathbb{M}(X, Y)$ the set of all operators from X to Y. We denote by $\overset{p}{\to}$ the point convergence on $\mathbb{M}(X, Y)$, by $\overset{unif}{\to}$ the uniform convergence and by $\overset{cont}{\to}$ the convergence with continuity (M. Agrisani and M. Clavelli [1]). Then $(\mathbb{M}(X, Y), \overset{p}{\to})$, $(\mathbb{M}(X, Y), \overset{unif}{\to})$ and $(\mathbb{M}(X, Y), \overset{cont}{\to})$ are L-spaces.

Remark 4.1. An L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces (in Perov' sense: $d(x, y) \in \mathbb{R}^m_+$, in Luxemburg-Jung' sense: $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$, $d(x, y) \in K$, K a cone in an ordered Banach space, $d(x, y) \in E$, E an ordered linear space with a notion of linear convergence, etc.), 2-metric spaces, $D - R-$spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details see Fréchet [11], Blumenthal [5] and I. A. Rus [29].

An important abstract concept is:

Definition 4.1 (Rus-Petruşel-Sîntămărian). Let $(X, \to)$ be an L-space. Then $T : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$
ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$
iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Example 4.4. Multivalued Reich type operators, multivalued Ciric type operators are examples of MWP operators.

Example 4.5 (Rus [28]). Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a closed multifunction for which there exist $\alpha, \beta \in \mathbb{R}_+$, with $\alpha + \beta < 1$ such that

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y)), \quad \text{for every } x \in X \quad \text{and every } y \in T(x).$$

Then T is a MWP operator.

Example 4.6 (Sîntămărian [33]). Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ for which there exists $\alpha \in \left]0, \frac{1}{2}\right[$ such that

$$H(T_1(x), T_2(y)) \leq a[D(x, T_1(x)) + D(y, T_2(y))], \quad \text{for each} \quad x, y \in X.$$

Then T_1 and T_2 are MWP operators.

Example 4.7 (Petruşel [21]). Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose that $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ satisfies the following assertions:

i) there exists $\alpha, \beta, \gamma \in \mathbb{R}_+$, with $\alpha + \beta + \gamma < 1$ such that:

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)) \quad \text{for all} \quad x, y \in \widetilde{B}(x_0; r)$$

ii) $\delta(x_0, T(x_0)) < \dfrac{1 - (\alpha + \beta + \gamma)}{1 - \gamma} r.$

Then T is a MWP operator.

Example 4.8 (Hadžic, see [14]). Let $(X, \mathcal{F})$ be a probabilistic metric space, $Y \in P(X)$ and $T : Y \to P(X)$. The multivalued operator T is said to be a probabilistic Nadler q-contraction if $q \in]0, 1[$ and for every $x, y \in X$, every $u \in T(x)$ and every $\delta > 0$ there exists $v \in T(y)$ such that for every $\varepsilon > 0$ we have

$$F_{u,v}(\varepsilon) \geq F_{x,y}\left(\frac{\varepsilon - \delta}{q}\right).$$

If T is a singlevalued operator then the notion of probabilistic Nadler q-contraction coincides with the notion of probabilistic q-contraction introduced by Sehgal and Bharucha-Reid.

Suppose $(X, \mathcal{F}, S)$ is a complete Menger space, S a t-norm of H-type, $Y \in P_{cl}(X)$ and $T : Y \to P_{cl}(Y)$ is a probabilistic Nadler q-contraction.

Then T is a MWP operator.

Example 4.9 (Czerwik [7], [8]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

i) $d(x, y) = 0$ if and only if $x = y$
ii) $d(x, y) = d(y, x)$
iii) $d(x, z) \leq s \cdot [d(x, y) + d(y, z)]$.

(X, d) is called a b-metric space. (see Czerwik [8])

From Czerwik [7] we have that if (X, d) is a complete b-metric space and $T : X \to P_{cl}(X)$ is a-Lipschitz with $a \in [0, s^{-1}[$, then T is a MWP operator.

Another important concept is:

Definition 4.2. Let $(X, \to)$ be an L-space. By definition, $f : X \to X$ is called a weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n \in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

Example 4.10. Let (X, d) be a complete metric space and $f : X \to X$ an orbitally continuous operator such that there is $a \in]0, 1[$ with the property $d(f(x), f^2(x)) \leq a \cdot d(x, f(x))$, for each $x \in X$. Then f is a WPO.

Example 4.11. Let (X, d) be a complete metric space, $f : X \to X$ an orbitally continuous operator and $\varphi : X \to \mathbb{R}_+$. We suppose that f satisfies the Caristi condition with respect to φ. Then f is a WPO.

In I. A. Rus [29] the basic theory of Picard and weakly Picard operators is presented. For the multivalued case see Petruşel [22]. For both settings see also [30].

Let $(X, \to)$ be an L-space and $F : X \to P(X)$. It is easy to see that if F admits a weakly Picard selection $f : X \to X$, then F is weakly Picard too.

If F is a weakly Picard mutivalued operator, in which conditions there exists a weakly Picard selection of it ?

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